Classification of abelian Schur groups

Grigory Ryabov

Sobolev Institute of Mathematics

Open Seminar
Shijiazhuang, May 19, 2026

S-rings

- G is a finite group, e is the identity of G.
- $\mathbb{Z}G$ is the integer group ring.

A subring $\mathcal{A} \subseteq \mathbb{Z}G$ is called an S-ring (Schur ring) over G if there exists a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ of G such that:

- $\{e\} \in \mathcal{S}$,
- $X \in \mathcal{S} \;\Rightarrow\; X^{-1} \in \mathcal{S}$,
- $\mathcal{A} = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : \; X \in \mathcal{S}\}$, where $\underline{X} = \sum_{x \in X} x$.

- The elements of $\mathcal{S}$ are called the basic sets of $\mathcal{A}$.
- The trivial S-ring $\mathcal{T}(G) = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : \; X \in \{\{e\}, G \setminus \{e\}\}\}$ if $G \neq \{e\}$.
- $\mathbb{Z}G = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : \; X \in \{\{g\} : \; g \in G\}\}$.
- The center $Z(\mathbb{Z}G)$ is an S-ring, basic sets are conjugacy classes.
- If $K \leq \mathrm{Aut}(G)$, then $\mathrm{Cyc}(K, G) = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : \; X \in \mathrm{Orb}(K, G)\}$ is a cyclotomic S-ring.

Applications of S-rings

Permutation group theory

- Studying permutation groups having a regular subgroup.
- Every primitive permutation group having a cyclic regular subgroup of composite order is 2-transitive (Schur, 1933).

Algebraic combinatorics

- S-rings are Cayley objects in the class of coherent configurations.
- Studying combinatorial Cayley objects, in particular, isomorphisms of them and constructing new Cayley objects.
- Cayley isomorphism problem (Babai-Frankl, 1978-...).
- Polynomial-time solution of the isomorphism problem for circulant graphs (Evdokimov-Ponomarenko, 2003, Muzychuk, 2004).

Representation theory

- Central S-rings are in one-to-one correspondence with supercharacters (Hendrickson, 2010).
- Studying representations of algebraic groups (Diaconis, Isaacs).

Schurian S-rings and Schur groups

- $G_{right} = \{x \mapsto xg,\ x \in G : \ g \in G\} \leq \mathrm{Sym}(G)$.
- $\mathrm{Orb}(K, G)$ is the set of all orbits of $K \leq \mathrm{Sym}(G)$ on G.

Theorem (Schur, 1933)

Let $K \leq \mathrm{Sym}(G)$ and $K \geq G_{right}$. Then
$V(K, G) = \mathrm{Span}_{\mathbb{Z}}\{\underline{X} : \ X \in \mathrm{Orb}(K_e, G)\}$ is an S-ring over G.

An S-ring $\mathcal{A}$ over G is called schurian if $\mathcal{A} = V(K, G)$ for some $K \leq \mathrm{Sym}(G)$ such that $K \geq G_{right}$.

- There exists a nonschurian S-ring over $E_{p^2} = C_p \times C_p$, where $p \geq 5$ is prime (Wielandt, 1964).

A finite group G is called a Schur group if every S-ring over G is schurian (Pöschel, 1974).

- A section of a Schur group is a Schur group.

Problem (Klin-Pöschel, 1974)

Determine all Schur groups.

Cyclic Schur groups

Theorem (Pöschel, 1974)

Let p be an odd prime. Cyclic p-groups are Schur groups and if $p \geq 5$, then a Schur p-group is cyclic.

- Cyclic 2-groups are Schur groups (Golfand-Najmark-Pöschel, 1985).
- Cyclic groups of order pq, where p and q are primes, are Schur (Klin-Pöschel, 1978)

Theorem (Evdokimov-Kovács-Ponomarenko, 2013)

Let $n \geq 1$ be an integer. A cyclic group of order n is a Schur group if and only if n belongs to one of the following families of integers:

$$p^k, pq^k, 2pq^k, pqr, 2pqr,$$

where p, q, r are primes and $k \geq 0$ is an integer.

Theorem (Leung-Man, 1996)

Let $\mathcal{A}$ be a nontrivial S-ring over a cyclic group. Then $\mathcal{A}$ is cyclotomic or a tensor or generalized wreath product of two S-rings.

Abelian Schur groups

- C_n and E_n are cyclic and elementary abelian groups of order n.

Theorem (Evdokimov-Kovács-Ponomarenko, 2016)

An elementary abelian noncyclic group of order n is a Schur group if and only if $n \in \{4, 8, 9, 16, 27, 32\}$.

Theorem (Evdokimov-Kovács-Ponomarenko, 2016)

An abelian Schur group which is neither cyclic nor elementary abelian belongs to one of the following families of groups:

- $C_2 \times C_{2^k}$, $C_{2p} \times C_{2^k}$, $E_4 \times C_{p^k}$, $E_4 \times C_{pq}$, $E_{16} \times C_p$,
- $C_3 \times C_{3^k}$, $C_6 \times C_{3^k}$, $E_9 \times C_q$, $E_9 \times C_{2q}$,

where p and q are distinct primes, $p \neq 2$, and $k \geq 1$ is an integer.

- The following groups are Schur groups:
  - $E_4 \times C_p$ (Evdokimov-Kovács-Ponomarenko, 2016);
  - $C_2 \times C_{2^k}$ (Muzychuk-Ponomarenko, 2015);
  - $C_3 \times C_{3^k}$ (R., 2017);
  - $E_9 \times C_q$ (Ponomarenko-R., 2018).

Abelian Schur groups

Theorem (R., 2026+)

The following groups are Schur groups:

- $E_4 \times C_{p^k}$, $E_4 \times C_{pq}$, $C_6 \times C_{3^k}$, $E_9 \times C_{2q}$,

where p and q are distinct primes, $p \neq 2$, and $k \geq 1$ is an integer.

Theorem (R., 2026+)

Let p be an odd prime. Then $C_{2p} \times C_{2^k}$ is a Schur group if and only if $k \leq 2$.

Theorem (R., 2026+)

Let p be an odd prime. Then $E_{16} \times C_p$ is a Schur group if and only if $p = 3$.

- All S-rings over the above Schur groups were characterized.
- New nonschurian S-rings over $C_{2p} \times C_8$ and $E_{16} \times C_p$, $p \geq 5$, were constructed.

Abelian Schur groups

Classification of abelian Schur groups

An abelian group is a Schur group if and only if it is isomorphic to one of the following groups:

- C_n, where $n \in \{p^k, pq^k, 2pq^k, pqr, 2pqr\}$, p, q, and r are primes, and $k \geq 0$ is an integer;
- E_n, where $n \in \{4, 8, 9, 16, 27, 32\}$;
- $C_2 \times C_{2^k}$, $C_4 \times C_{2p}$, $E_4 \times C_{p^k}$, $E_4 \times C_{pq}$, $E_{16} \times C_3$, where $p \neq 2$ and q are distinct primes and $k \geq 1$ is an integer;
- $C_3 \times C_{3^k}$, $C_6 \times C_{3^k}$, $E_9 \times C_q$, $E_9 \times C_{2q}$, where q is a prime and $k \geq 1$ is an integer.

Corollary

The Leung-Man theorem holds for every abelian Schur group.

Nonabelian Schur groups

- Every group of order at most 15 is a Schur group. In particular, there are nonabelian Schur groups (computer calculations, Fiedler, 1998).

Theorem (Ponomarenko-Vasil'ev, 2014)

Every Schur group G is metabelian and $\pi(|G|) \leq 7$.

- Using the results of Ponomarenko-Vasil'ev (2014), Muzychuk-Ponomarenko (2015), Ryabov (2015), we prove the theorem below.

Theorem

A nonabelian nilpotent Schur group is isomorphic to one of the groups below:

- Q_8, $D_8 * C_4$, D_{2^k}, $k \geq 3$,
- $Q_8 \times C_p$, where $p \geq 11$ is a prime such that $p \equiv (-1) \mod 12$.

Moreover, Q_8, $D_8 * C_4$, D_{2^k}, $3 \leq k \leq 5$, are Schur groups.

Dihedral Schur groups

- A semidirect product of an abelian group and a cyclic group of order 2, where the nontrivial element of the latter inverts every element of the former, is called a generalized dihedral group

Theorem (R., 2025)

Every generalized dihedral Schur group is dihedral.

- $\pi(n)$ is the number of pairwise distinct prime divisors of an integer n

Theorem (R., 2025)

Let $n \geq 3$ be a positive integer. Suppose that a dihedral group of order $2n$ is a Schur group. Then $|\pi(n)| \leq 3$. Moreover,

- if $|\pi(n)| = 2$, then n is a product of a prime and an odd prime power or $n = 12$,
- if $|\pi(n)| = 3$, then n is a product of three primes.

- If $p \in \pi(n)$ and $p \equiv 3 \mod 4$, then $p = 3$ or $n \in \{7, 11, 14, 22\}$.
- If $p, q \in \pi(n)$ are odd and $p < q$, then $q \not\equiv 1 \mod p$.

Dihedral Schur groups

Question

Does an infinite family of nonabelian Schur groups exist?

- The largest previously known nonabelian Schur group has order 63.

Theorem (R., 2025)

Let p be a prime. If p is a Fermat prime or $p = 4q + 1$, where q is a prime, then D_{2p} is a Schur group.

- The largest known Fermat prime is 65537.
- There are infinitely many primes $p = 4q + 1$ modulo some famous (and widely believed) number-theoretical conjectures (Dickson, generalized Hardy-Littlewood).
- A key ingredient of the proof is nonexistence of a difference set in C_p, where p is a Fermat prime or $p = 4q + 1$.
- If $p \equiv 3 \mod 4$ and $p > 11$ or $p = 4t^2 + 1$, where $t \geq 3$ is an odd integer, then C_p has a nontrivial difference set and D_{2p} is not a Schur group.

Frobenius Schur groups

- A group $G \leq \mathrm{Sym}(\Omega)$ is Frobenius if G is transitive nonregular and there is no nontrivial element of G fixing more than one point.
- $G = H \rtimes K$, where H is a normal regular subgroup of G (kernel) and K is a one-point stabilizer (complement).
- $\mathcal{P}$ is the set of all integers of one of the forms $p^k,\ pq^k, 2pq^k,\ pqr,\ 2pqr$, where p, q, and r are primes and $k \geq 1$.
- $\mathcal{P}_0$ is the set of all odd integers from $\mathcal{P}$.

Theorem (R., 2025)

Let G be a Frobenius Schur group. Then G is isomorphic to one of the following groups:

- $C_n \rtimes C_m$, where $n \in \mathcal{P}_0$ and $m \in \mathcal{P}$;
- $(E_4 \times C_{p^k}) \rtimes C_3$, $(E_4 \times C_{pq}) \rtimes C_3$, where p and q are odd primes and $k \geq 1$;
- $E_4 \rtimes C_3$, $E_{27} \rtimes C_{13}$, $E_{32} \rtimes C_{31}$.